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Theorem of Expended Power and Finite Element Formulations: Hamiltonian Mechanics Framework

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Within the Hamiltonian mechanics framework described here, we have developed in this paper a general numerical discretization via the finite element method for continuum-elastodynamics applications directly stemming from the theorem of expended power involving a built-in scalar function: namely, the autonomous Hamiltonian ($\mathcal{H}(\mathbf{p},\mathbf{q})$: $T^*Q \rightarrow \mathbb{R}$). To provide alternative viewpoints and new avenues, a differential formulation is proposed here rather than resorting to classical variational formulations in which, historically, most finite element developments dealing with elastodynamics have been traditionally described. We place particular emphasis upon the notion that the scalar function that represents the autonomous Hamiltonian of the continuous-body dynamical systems can be a crucial mathematical function and physical quantity that is a constant of motion in conservative systems; this is in contrast to vector quantities customarily employed with Newton-based formulations for which physical quantities are more difficult to associate. As such, the scalar representation enables one to readily capitalize on theorems such as Noether's to establish symmetry properties so that proof of satisfaction of the discrete system as that of the continuous system can be readily established, unlike traditional practices emanating from Newton-based representations. The proposed concepts emanating from the theorem of expended power inherently involving the scalar function (namely, the Hamiltonian) naturally embody the weak form in space that can be described by a discrete Hamiltonian differential operator, and integrating over a given time interval yields the weighted residual in time statement. Thereby, a novel yet simple space-discrete Hamiltonian formulation proposed here only needs to employ the discrete Hamiltonian differential operator, which provides new and alternate avenues and directly yields the semidiscrete ordinary differential equations in time that can be readily shown to preserve the same physical attributes as the continuous systems for continuum-dynamical applications. Theoretical formulations are shown for selected structural elements for illustration of the basic concepts.

I. Introduction

IT APPEARS that the inclusion of the variations in physics has been the focus of philosophical controversies and misinterpretations, due to the perception that the variational calculus seems to bring a purpose to the flow of natural events [1]. The differential calculus [Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) independently developed the differential calculus and Newton first applied it to physics] or the variational calculus [Leonhard Paul Euler (1707–1783) and Joseph-Louis Lagrange (1736–1813) developed the variational calculus in the 18th century, often called the calculus of variations [2], and they used it extensively to derive the differential equations specifically for mechanical systems] is concerned with the infinitesimal changes of the physical quantities.

For instance, Hamilton's principle in dynamics [1,3–7], which is one of the variational principles, is well known to be a fundamental axiom in mechanics and science. It claims that extremizing the action as a functional, first defined and referred to as the principal function by Hamilton [8–10], leads to Lagrange's equations of motion as governing equations for N-body systems, as well as its ability to derive both the governing equations and boundary conditions in continuum-elastodynamics that are continuous in space and time. However, it has been well known that Hamilton's principle possesses a logic inconsistency, as pointed out in the literature [11–17], regarding forcing the other end point in time to be also prescribed. In this regard, we have previously presented via Hamilton's law of varying action a detailed formulation leading to consistent numerical discretization with focus on space discretization [18] that is noteworthy, although it is also a variational approach.

Ludwig Eduard Boltzmann (1844–1906) classified holonomic systems into scleronomic (meaning rigid) and rheonomic (meaning flow) [1]. In the rheonomic systems, the introduction of the variation enables one to eliminate the time partial-derivative term. The direction of the differential in the rheonomic systems is different from that of the variation, due to the presence of the partial derivative of any physical quantity depending on time explicitly, with respect to time. Literature cites certain advantages of the variational calculus over the differential calculus in the rheonomic systems; but this has to be further explored. Any advantages disappear in scleronomic systems. Note that the variation is imaginary and a mathematical abstract quantity. Most engineering and science applications fall into the class of the scleronomic systems [19]. In the scleronomic systems, the differential can replace the variation ([20], page 14). Therefore, for engineering and science applications that are scleronomic, the variational approach is not particularly necessary to obtain governing equations or to find approximate solutions, and alternatives such as the differential formulations can also be employed; this is the main point. In holonomic-scleronomic systems, the Hamiltonian becomes autonomous. In particular, the areas of solid mechanics or elastodynamics or continuum mechanics to which the finite element formulations are usually applied are generally scleronomic. As a consequence, variational formulations are not indispensable to such applications, and differential forms are also ideal for several of these situations, as described in this paper.

Amalie Emmy Noether (1882–1935) recognized the presence of symmetry in physics. The symmetry in physics corresponds to invariance properties of the scalar functions such as the Lagrangian and the Hamiltonian of dynamical systems [21–24]. In view of Noether's theorem, we have the statement that the spatial translational invariance of the Lagrangian in the configuration space gives

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conservation of linear momentum, the rotational invariance gives conservation of angular momentum, and the autonomous Lagrangian has invariance in time. Likewise, the Hamiltonian in the phase space is known to have the same invariance properties [25]. *This is the so-called Hamiltonian version of Noether's theorem* [26], *which is of focus and interest here.* Not only in the infinite-dimensional continuous-body system, but also in the finite-dimensional continuous-body system, we have the invariance properties of the space-discrete Hamiltonian, as discussed later in this work.

In consideration of Noether's theorem, we wish to state that conservation laws [7] play an important role in obtaining governing equations in physics and mechanics. Because of Noether's celebrated work [27–30], there exist conservation laws for three important physical quantities: linear momentum, angular momentum, and total energy. The conservation laws can be extended to the balance laws [31] for mechanical systems such as the linear momentum balance law, the angular momentum balance law, and the energy balance law in continuum mechanics [27,28,31]. Newton's equations of motion for N-body dynamical systems or Cauchy's equations of motion for continuous-body dynamical systems can be obtained from the notion of the linear momentum balance law. The D'Alembert–Lagrange principle [6] or the generalized principle of D'Alembert [29] or the Lagrangian form of D'Alembert's principle [20] results from the conceptual amalgamation of the linear momentum balance law and the principle of virtual work. Note that Lagrange's equations of motion can be derived from the D'Alembert–Lagrange principle; furthermore, Hamilton's equations of motion can be deduced from Lagrange's equations of motion and invoking Legendre transformation. As a consequence, the governing equations including Lagrange's equations of motion, Hamilton's equations of motion, and Cauchy's equations of motion have been obtained from the notion of the linear momentum balance law. However, we abandon this approach. As an alternative, we employ the notion of the total energy balance law, which results in the theorem of expended power, as the fundamental axiom.

In both physics and mechanics, variational principles [1,29,30,32] have been regarded as an important role for deriving the governing differential equations. For instance, Hamilton's principle has been employed in dynamics, in the electromagnetic and gravitational fields, and even in quantum mechanics. Furthermore, we can consider Fermat's principle of least time [3] as leading to Snell's law in optics and several variational principles in continuum-statics such as Hu–Washizu's variational principle Hellinger–Reissner variational principle, Maupertuis's principle of least action [3], Jacobi and Lagrange's principle of least action [4] in dynamics again, and Gurtin's variational principles [11,12] for intial-value problems such as wave and transient heat transfer in lieu of Hamilton's principle and the like. All of these variational principles have been considered in engineering and science as indispensable branches of mathematics with which to derive the differential equations (namely, the Euler–Lagrange equations).

Other than variational principles to obtain governing equations, the variational methods [30,33–35] have been used to find approximate solutions of the governing equations for obtaining a system of algebraic equations. The methods that pertain to this category include the Rayleigh–-Ritz method, the principle of virtual work, the Galerkin weighted residual method, the Kantorovitch method, the Trefftz method, and so on. These methods are sometimes called direct methods to find approximate solutions. Regardless of whether the discipline is the variation principle or the variational method, the notion of the variation should be invoked in the associated mathematical formulation. A question arises at this point: what about the differential instead of the variation? Note that the variation is imaginary but the differential is real.

The classical Ritz method requires a functional to be given. The functional for the Ritz method can be found in static/steady problems, but it does not readily exist in initial-boundary-value problems (transient/dynamical problems). Therefore, the classical Ritz method is not readily applicable to transient problems such as transient heat transfer, continuum dynamics, and so on. The Ritz method uses the functional that allows a weak continuity condition for trial functions satisfying essential boundary conditions. Because both the governing equations and natural boundary conditions are contained implicitly in the functional in the Ritz solution that operates on the functional corresponding to the problem, there is a mechanism by which the errors in the satisfaction of the governing equations and natural boundary conditions are minimized when substituting the trial functions into the functional to discretize the space ([36], page 108; [37], page 136; [38], page 61; and [39], page 74).

On the other hand, the classical Bubnov–Galerkin method requires a strong form for boundary-value problems with homogeneous boundary conditions to be given. The method requires one to choose two admissible functions such as trial and test functions, in contrast to the Ritz method discussed previously. These functions should satisfy both essential and natural boundary conditions. Furthermore, the continuity condition for these admissible functions is stronger than the functional in the Ritz method. Unless the differential operator is symmetric, the resulting coefficient matrix is not guaranteed to be symmetric. It is very hard to find suitable admissible functions satisfying natural boundary conditions. Once the natural boundary conditions are given, on the other hand, the so-called modified Bubnov–Galerkin method and extensions to the finite element method derive a weak form by integration by parts, and natural boundary conditions can be then enforced into the weak form. The weak form by integration by parts allows a weaker continuity condition for both trial and test functions than in the Ritz method.

Other than concepts emanating from Newtonian mechanics framework first, not much is described for finite element developments in elastodynamics applications that use representations involving scalar functions with generalized coordinates as a starting point. In the finite element communities, much attention [40,41] was first paid to the action as a functional involving the scalar function (namely, the Lagrangian) to mostly develop time integration schemes. The Hamiltonian structure of the Hamiltonian systems has been fully discussed in the works [24,31,42]. Specifically, regarding the work [42], a coordinate-independent covariant form of Cauchy's equations of motion can be represented in the material/spatial/convective description. In their derivation, the body forces are neglected. Although their representations are in covariant form [[42], see Eqs. (2.91) and (2.94)], one form convected density function obtained as the tensor product of a convected one form with the convected volume element other than the scalar function such as the Hamiltonian density function, which we have defined and used here, are described in their work. Furthermore, all of these works mentioned previously are not aimed to develop the finite element formulation with a scalar function, such as the Hamiltonian, under the Hamiltonian mechanics framework. In contrast to Newtonian mechanics and traditional practices, not much is discussed on how to foster numerical spatial discretizations for frameworks that are associated with the concept of scalar representations with generalized coordinates. Nonetheless, recognizing the simple fact that indeed the nodal variables representing the degrees of freedom in finite element computations are by themselves representative of generalized coordinates, the Hamiltonian mechanics framework, which is described herein, is indeed also ideal for enabling straightforward numerical discretizations such as with the finite-element-type methods.

In this exposition, we focus upon the following:

1) The basic issue with the conventional approaches for continuum dynamics that are Newton-based is that there is no notion of a *scalar function* (say, the Hamiltonian). It mostly relies upon numerical discretization of the governing equations with dependent variables in terms of vectors quantities such as displacement, velocity, acceleration, etc. The notion of the Hamiltonian in the phase space, however, requires first-order system representations. The Hamiltonian we use herein is the kinetic energy plus the potential energy for N-body systems and the kinetic energy plus the total potential energy for continuous-body systems. Such systems are referred to as the Hamiltonian systems, which are said to be scleronomic. Within the Hamiltonian framework, we describe a new alternative finite element formulation in continuum dynamics emanating from the theorem of

expended power. The notion of variational calculus is discarded, and we focus attention upon differential calculus instead. The theorem of expended power reduces to the rate of the Hamiltonian. This proposition is backed up by the observation that the Hamiltonian we employ in phase space possesses translational and rotational invariance properties in space, and the autonomous Hamiltonian has symmetry under translation in time. Note that such symmetries exist in the Hamiltonian systems [24], regardless of whether the dynamical systems are of finite or infinite dimension. To ensure that the discretized equations resulting from the finite element method inherit the same physics from the continuous governing equations in space and time, it is not readily straightforward to develop such proofs via traditional practices, because no scalar function readily exists to capitalize on theorems such as Noether's [21–24].

2) The differential calculus as an alternative to the variational calculus is employed here not only to derive the governing differential equations, but also to enable approximate solutions for the differential equations. We do not resort to the variational principle or the variational method involving the notion of variation such as the virtual displacement or the virtual work.

3) We focus upon the energy balance law as an alternative to the linear momentum balance law. It is our belief that the governing equations can be extracted from any conservation law.

Traditional practices that are employed via a hybrid weighted residual form leading to the weak form or the principle of virtual work are strictly statements of the weak form in space only. In contrast, we demonstrate that the total time derivative of the Hamiltonian, which is derived from the theorem of expended power, consequently allows the weaker continuity condition in space for the trial function for the spatial discretization, resulting in a space-discrete Hamiltonian differential operator. Finally, we demonstrate a novel yet simple space-discrete Hamiltonian formulation that directly yields the semidiscrete ordinary differential equations in time, which can be readily shown to preserve the same physical attributes as the continuous systems for continuum-dynamical applications, unlike traditional practices. Accordingly, we simply discretize the total time derivative of the Hamiltonian in space by only admissible trial functions, and we need not resort to traditional practices and approaches such as the hybrid weighted residual form, involving both admissible trial and test functions leading to constructing the weak form in continuum-elastodynamics to enable numerical discretizations; this is the main point. In addition, after discretization in space, the time integral form of the total time derivative of the Hamiltonian naturally takes the representation of the hybrid weighted residual form in time (that is, in the sense of finite elements in time), which then motivates one to conduct time-weighted residual formulations [43] or time-finite-element formulations [16] for the design of appropriate time-marching integration schemes.

II. Theorem of Expended Power for Continuous-Dynamical Systems

In elastodynamics, for a continuous body (which is a simply connected, bounded, and open set $\mathcal{B} \subset \mathbb{R}^3$ with a piecewise smooth boundary open set $\partial\mathcal{B}$ in the reference configuration), it is often postulated that infinite particles construct the body; the particles are not considered to have concentrated mass. The particles can be defined as mathematical quantities, sometimes called material points, $\mathbf{X} \in \mathcal{B} \subset \mathbb{R}^3$. The smooth motion of the body may be described as a one-to-one mapping, $\mathbf{X} = \boldsymbol{\varphi}(\mathbf{X}, t)\ \forall\ t \in \mathcal{T} = [t_1, t_2]$. The space occupied by a set of all configurations in the dynamical motion is called the configuration space $\mathcal{C} = \{\boldsymbol{\varphi}\colon \mathcal{B} \times \mathcal{T} \to \mathcal{S} \subset \mathbb{R}^3\}$, where $\mathcal{B}$ stands for the reference configuration, and $\mathcal{S}$ denotes the current configuration. The configuration space is known to be a smooth infinite-dimensional manifold [25].

Let us assume that the continuous body is not subject to prescribed distributed body or surface couples. As a rate at which the external surface tractions and body forces are doing actual work, the power input $\mathcal{P}_{\text{input}}$ is defined as

$$\mathcal{P}_{\text{input}} = \int_{\mathcal{B}} \rho_0 \mathbf{B}(\mathbf{X}, t) \cdot \dot{\boldsymbol{\varphi}}(\mathbf{X}, t)\, \mathrm{d}V + \int_{\partial\mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}, t) \cdot \dot{\boldsymbol{\varphi}}(\mathbf{X}, t)\, \mathrm{d}A \tag{1}$$

where both $\mathrm{d}A$ and $\mathrm{d}V$ denote the Lebesgue measures on the prescribed-traction boundary $\partial\mathcal{B}_\sigma$ and in the body $\mathcal{B}$, $\rho_0(\mathbf{X})\colon \mathcal{B} \to \mathbb{R}$ denotes the referential density, $\bar{\mathbf{T}}(\mathbf{X}, t)\colon \mathcal{B} \to \mathbb{R}^3$ stands for the prescribed traction on the boundary $\partial\mathcal{B}_\sigma$, and $\mathbf{B}(\mathbf{X}, t)\colon \mathcal{B} \to \mathbb{R}^3$ is the body force in the body $\mathcal{B}$. Note that the preceding external forces are not dead loads. By invoking Cauchy's law $\bar{\mathbf{T}}(\mathbf{X}, t) = \mathbf{P}(\mathbf{X}, t) \cdot \hat{\mathbf{N}}(\mathbf{X}, t)$ and Gauss's theorem, the preceding equation can rewritten as

$$\mathcal{P}_{\text{input}} = \int_{\mathcal{B}} \rho_0 \mathbf{B}(\mathbf{X}, t) \cdot \dot{\boldsymbol{\varphi}}(\mathbf{X}, t)\, \mathrm{d}V + \int_{\mathcal{B}} \nabla \cdot (\mathbf{P}(\mathbf{X}, t) \cdot \dot{\boldsymbol{\varphi}})\, \mathrm{d}V \tag{2}$$

where the first Piola–Kirchhoff stress tensor is denoted by $\mathbf{P}(\mathbf{X}, t)$ [28,31] in the reference configuration. It should be noted that the preceding equation is obtained by imposing the natural boundary condition on Eq. (1). Because of the product rule, we have the relation

$$\nabla \cdot (\mathbf{P} \cdot \dot{\boldsymbol{\varphi}}) = (\nabla \cdot \mathbf{P}) \cdot \dot{\boldsymbol{\varphi}} + \mathbf{P} \cdot\cdot (\nabla \dot{\boldsymbol{\varphi}}) = (\nabla \cdot \mathbf{P}) \cdot \dot{\boldsymbol{\varphi}} + \mathbf{P} \cdot\cdot \dot{\mathbf{F}} \tag{3}$$

where $\dot{\mathbf{F}}$ denotes the rate of the deformation gradient tensor $\mathbf{F}$. By use of Eq. (3), the power input can be expressed as

$$\mathcal{P}_{\text{input}} = \int_{\mathcal{B}} \{\nabla \cdot \mathbf{P} + \rho_0 \mathbf{B}(\mathbf{X}, t)\} \cdot \dot{\boldsymbol{\varphi}}(\mathbf{X}, t)\, \mathrm{d}V + \int_{\mathcal{B}} \mathbf{P} \cdot\cdot \dot{\mathbf{F}}\, \mathrm{d}V \tag{4}$$

After imposing Cauchy's equations of motion on the preceding equation, we have the power input as shown:

$$\mathcal{P}_{\text{input}} = \int_{\mathcal{B}} \rho_0 \ddot{\boldsymbol{\varphi}} \cdot \dot{\boldsymbol{\varphi}}(\mathbf{X}, t) \mathrm{d}V + \int_{\mathcal{B}} \mathbf{P} \cdot\cdot \dot{\mathbf{F}} \mathrm{d}V \tag{5}$$

The first term on the right-hand side in Eq. (5) corresponds to the rate of the kinetic energy of the system. The second is referred to as the stress power [27]. We observe that the power input is equivalent to the stress power plus the rate of the kinetic energy. Consequently, in view of Eqs. (1) and (5), we arrive at the theorem of expended power ([44], page 110):

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} + \int_{\mathcal{B}} \mathbf{P} \cdot\cdot \dot{\mathbf{F}}\, \mathrm{d}V = \int_{\mathcal{B}} \rho_0 \mathbf{B}(\mathbf{X}, t) \cdot \dot{\boldsymbol{\varphi}}\, \mathrm{d}V + \int_{\partial\mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}, t) \cdot \dot{\boldsymbol{\varphi}}\, \mathrm{d}A \tag{6}$$

Note that the preceding equation has been obtained by the use of both governing equations (Cauchy's equations) and natural boundary conditions (Cauchy's law). Of critical importance is that the preceding equation holds regardless of whether the system is mechanical or thermomechanical. In view of Eqs. (3) and (5), the theorem of expended power can be rewritten as the Bubnov–Galerkin weighted residual form as follows:

$$\int_{\mathcal{B}} (\rho_0 \ddot{\boldsymbol{\varphi}} - \nabla \cdot \mathbf{P} - \rho_0 \mathbf{B}) \cdot \dot{\boldsymbol{\varphi}} \mathrm{d}V + \int_{\partial\mathcal{B}_\sigma} (\mathbf{P}(\mathbf{X}, t) \cdot \hat{\mathbf{N}}(\mathbf{X}, t) - \bar{\mathbf{T}}(\mathbf{X}, t)) \cdot \dot{\boldsymbol{\varphi}} \mathrm{d}A = 0 \tag{7}$$

Thus, we observe that the theorem of expended power results in the Bubnov–Galerkin weighted residual form, in which the weighting function is the generalized velocity or the differential of position (or the differential of displacement). Note that the preceding equation can be rewritten as

$$\int_{\mathcal{B}} (\rho_0 \ddot{\boldsymbol{\varphi}} - \nabla \cdot \mathbf{P} - \rho_0 \mathbf{B}) \cdot \mathrm{d}\boldsymbol{\varphi}\, \mathrm{d}V + \int_{\partial\mathcal{B}_\sigma} (\mathbf{P}(\mathbf{X}, t) \cdot \hat{\mathbf{N}}(\mathbf{X}, t) - \bar{\mathbf{T}}(\mathbf{X}, t)) \cdot \mathrm{d}\boldsymbol{\varphi}\, \mathrm{d}A = 0 \tag{8}$$

In the preceding equation, the weighting function becomes the differential of position or displacement. In view of Eq. (7) or Eq. (8),

both governing equations and natural boundary conditions are implicitly contained in the theorem of expended power given in Eq. (6). Therefore, the spatial discretization of Eq. (6) means that the residual errors in the satisfaction of the governing equations and natural boundary conditions are minimized when Eq. (7) is regarded as a mechanism by which the weighted residual average can vanish.

In the case of no entropy production or isothermal process, the internal dissipation $\mathcal{D}_{\text{int}}$ can be given by

$$\mathcal{D}_{\text{int}} = \mathbf{P} \cdot\cdot \dot{\mathbf{F}} - \rho_0 \frac{\partial \Psi}{\partial t} \geq 0 \tag{9}$$

where $\Psi(\mathbf{F})$ denotes the Helmholtz free energy per unit mass in homogeneous materials. In the thermodynamics process, the material for which the mechanical behavior shows no internal dissipation is called the perfect material ([45], page 303). Then we have the equality equation:

$$\mathcal{D}_{\text{int}} = \mathbf{P} \cdot\cdot \dot{\mathbf{F}} - \rho_0 \frac{\partial \Psi}{\partial t} = 0 \tag{10}$$

It is said that the first Piola–Kirchhoff stress tensor is the thermodynamics force work conjugate to the deformation gradient tensor [28,46] as follows:

$$\mathbf{P} = \rho_0 \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} \tag{11}$$

In the reversible process (thermal effects are neglected and the entropy is not produced in the process), the preceding equation also holds: namely, for mechanical systems only, we have

$$\mathbf{P} \cdot\cdot \dot{\mathbf{F}} = \rho_0 \frac{\partial \Psi}{\partial t} = \frac{\partial W}{\partial t} \tag{12}$$

where W stands for the elastic potential energy or the stored function energy or the strain energy density. In view of Eq. (10), regardless of whether the system is thermomechanical or mechanical, the theorem of expended power shown in Eq. (6) can be written as

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} + \int_{\mathcal{B}} \rho_0 \frac{\partial \Psi}{\partial t}\, \mathrm{d}V = \int_{\mathcal{B}} \rho_0 \mathbf{B}(\mathbf{X}, t) \cdot \dot{\boldsymbol{\varphi}}(\mathbf{X}, t)\, \mathrm{d}V + \int_{\partial \mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}, t) \cdot \dot{\boldsymbol{\varphi}}(\mathbf{X}, t)\, \mathrm{d}A \tag{13}$$

Equation (13) indicates that the summation of the rate of the kinetic energy and the rate of the internal potential energy equals the power input. Note that the external forces in Eq. (13) are not dead loads and there does not exist the external potential energy. Equation (13) is true for nonconservative systems, and we obtain

$$\frac{\mathrm{d}\mathcal{E}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}})}{\mathrm{d}t} = \mathcal{P}_{\text{input}}, \qquad \mathcal{E}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}) = \mathcal{K}(\dot{\boldsymbol{\varphi}}) + \mathcal{U}_{\text{int}}(\boldsymbol{\varphi}) \tag{14}$$

where the kinetic energy $\mathcal{K}(\dot{\boldsymbol{\varphi}})$ and the internal potential energy $\mathcal{U}_{\text{int}}(\boldsymbol{\varphi})$ are defined by

$$\mathcal{K}(\dot{\boldsymbol{\varphi}}) = \frac{1}{2} \int_{\mathcal{B}} \rho_0 \dot{\boldsymbol{\varphi}} \cdot \dot{\boldsymbol{\varphi}}\, \mathrm{d}V, \qquad \mathcal{U}_{\text{int}}(\boldsymbol{\varphi}) = \int_{\mathcal{B}} \rho_0 \Psi(\mathbf{F})\, \mathrm{d}V \tag{15}$$

where the kinetic energy is quadratic in generalized momentum.

Let us assume that a continuous-dynamical system is subject to dead loads; therefore, the system is conservative and the external potential energy exists. Then instead of Eq. (14), we have

$$\frac{\mathrm{d}\mathcal{E}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}})}{\mathrm{d}t} = 0, \qquad \mathcal{E}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}) = \mathcal{K}(\dot{\boldsymbol{\varphi}}) + \mathcal{U}(\boldsymbol{\varphi})$$
$$\mathcal{U}(\boldsymbol{\varphi}) = \mathcal{U}_{\text{int}}(\boldsymbol{\varphi}) + \mathcal{U}_{\text{ext}}(\boldsymbol{\varphi}) \tag{16}$$

where $\mathcal{U}(\boldsymbol{\varphi})$ denotes the total potential energy, and the external potential energy is given by

$$\mathcal{U}_{\text{ext}}(\boldsymbol{\varphi}) = -\int_{\partial \mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}) \cdot \boldsymbol{\varphi}(\mathbf{X}, t)\, \mathrm{d}A - \int_{\mathcal{B}} \rho_0 \mathbf{B}(\mathbf{X}) \cdot \boldsymbol{\varphi}(\mathbf{X}, t)\, \mathrm{d}V \tag{17}$$

Via introducing the generalized momentum, the total energy given in Eq. (16) can be converted to the Hamiltonian as follows:

$$\mathcal{E}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}) = \mathcal{K}(\dot{\boldsymbol{\varphi}}) + \mathcal{U}(\boldsymbol{\varphi}) \Rightarrow \mathcal{K}(\boldsymbol{\phi}) + \mathcal{U}(\boldsymbol{\varphi}) = \mathcal{H}(\boldsymbol{\phi}, \boldsymbol{\varphi}) \tag{18}$$

Finally, instead of Eq. (16), we have

$$\frac{\mathrm{d}\mathcal{H}(\boldsymbol{\phi}, \boldsymbol{\varphi})}{\mathrm{d}t} = 0, \qquad \mathcal{H}(\boldsymbol{\phi}, \boldsymbol{\varphi}) = \mathcal{K}(\boldsymbol{\phi}) + \mathcal{U}(\boldsymbol{\varphi}) \tag{19}$$

This is the theorem of expended power for the autonomous Hamiltonian system within the Hamiltonian mechanics framework. Note that the dependent variables are canonical coordinates in the phase space.

III. Hamiltonian Finite Element Formulation for Continuous-Dynamical Systems

Consider a Hamiltonian system [31,42] in which the Hamiltonian is preserved in time as a constant of motion. Then the continuous-body dynamical system becomes scleronomic, and the power input for scleronomic conservative systems (the external forces are dead loads) is given by

$$\mathcal{P}_{\text{input}} = \int_{\mathcal{B}} \rho_0 \mathbf{B}(\mathbf{X}) \cdot \dot{\boldsymbol{\varphi}}\, \mathrm{d}V + \int_{\partial \mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}) \cdot \dot{\boldsymbol{\varphi}}\, \mathrm{d}A = -\frac{\mathrm{d}\mathcal{U}_{\text{ext}}}{\mathrm{d}t} \tag{20}$$

where the surface tractions and body forces are dead loads. Note that the preceding body forces and tractions are prescribed. Therefore, as defined in the preceding equation, external potential energies do exist due to dead loads. The autonomous Lagrangian $\mathcal{L}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}})$: $T\mathcal{C} \to \mathbb{R}$, the kinetic energy minus the total potential energy, with the definition of the tangent bundle (velocity phase space)

$$T\mathcal{C} := \{(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}) | \boldsymbol{\varphi} \in \mathcal{C} \quad \text{and} \quad \dot{\boldsymbol{\varphi}} \in T_{\boldsymbol{\varphi}}\mathcal{C}\} \tag{21}$$

is defined by

$$\mathcal{L}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}) = \int_{\mathcal{B}} \bar{\mathcal{L}}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}, \mathbf{F})\, \mathrm{d}V + \int_{\partial \mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}) \cdot \boldsymbol{\varphi}(\mathbf{X}, t)\, \mathrm{d}A$$
$$\mathcal{L}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}) = \mathcal{K}(\dot{\boldsymbol{\varphi}}) - \mathcal{U}(\boldsymbol{\varphi}) \tag{22}$$

where the Lagrangian density function $\bar{\mathcal{L}}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}, \mathbf{F})$ is defined as

$$\bar{\mathcal{L}}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}, \mathbf{F}) = \tfrac{1}{2} \rho_0 \dot{\boldsymbol{\varphi}} \cdot \dot{\boldsymbol{\varphi}} - \rho_0 \Psi(\mathbf{F}) + \rho_0 \mathbf{B}(\mathbf{X}) \cdot \boldsymbol{\varphi} \tag{23}$$

Let us introduce a new dependent variable, the canonical momentum, which belongs to the cotangent space $\boldsymbol{\phi} \in T^*_{\boldsymbol{\varphi}}\mathcal{C}$:

$$\boldsymbol{\phi} = \frac{\partial \bar{\mathcal{L}}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}, \mathbf{F})}{\partial \dot{\boldsymbol{\varphi}}} = \rho_0 \dot{\boldsymbol{\varphi}}(\mathbf{X}, t) \tag{24}$$

Then we may define the Hamiltonian density function via the Legendre transformation:

$$\bar{\mathcal{H}}(\boldsymbol{\phi}, \boldsymbol{\varphi}, \mathbf{F}) = \boldsymbol{\phi} \cdot \dot{\boldsymbol{\varphi}} - \bar{\mathcal{L}}(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}, \mathbf{F}) \tag{25}$$

where the Hamiltonian density function in view of Eq. (23) is given by

$$\bar{\mathcal{H}}(\boldsymbol{\phi}, \boldsymbol{\varphi}, \mathbf{F}) = \frac{1}{2\rho_0} \boldsymbol{\phi} \cdot \boldsymbol{\phi} + \rho_0 \Psi(\mathbf{F}) - \rho_0 \mathbf{B}(\mathbf{X}) \cdot \boldsymbol{\varphi} \tag{26}$$

In contrast to the autonomous Lagrangian, which is defined on the tangent bundle, the autonomous Hamiltonian is defined on the cotangent bundle (phase space):

$$T^*\mathcal{C} = \{(\boldsymbol{\phi}, \boldsymbol{\varphi}) | \boldsymbol{\varphi} \in \mathcal{C} \text{ and } \boldsymbol{\phi} \in T^*_{\varphi}\mathcal{C}\} \tag{27}$$

Thus, the autonomous Hamiltonian $\mathcal{H}(\boldsymbol{\phi}, \boldsymbol{\varphi})$: $T^*\mathcal{C} \to \mathbb{R}$, for which the domain is an infinite-dimensional space, can be defined as

$$\mathcal{H}(\boldsymbol{\phi}, \boldsymbol{\varphi}) = \int_{\mathcal{B}} \bar{\mathcal{H}}(\boldsymbol{\phi}, \boldsymbol{\varphi}, \mathbf{F})\, \mathrm{d}V - \int_{\partial \mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}) \cdot \boldsymbol{\varphi}(\mathbf{X}, t)\, \mathrm{d}A \quad (28)$$

where the external forces are dead loads. Then the total time derivative of the autonomous Hamiltonian can be written as

$$\frac{\mathrm{d}\mathcal{H}(\boldsymbol{\phi}, \boldsymbol{\varphi})}{\mathrm{d}t} = \int_{\mathcal{B}} \frac{\mathrm{d}\bar{\mathcal{H}}(\boldsymbol{\phi}, \boldsymbol{\varphi}, \mathbf{F})}{\mathrm{d}t}\, \mathrm{d}V - \int_{\partial \mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}) \cdot \dot{\boldsymbol{\varphi}}(\mathbf{X}, t) = 0 \quad (29)$$

In view of the preceding equation, the autonomous Hamiltonian for conservative continuous-dynamical systems can be regarded as a constant of motion. By use of the chain rule, we have

$$\frac{\mathrm{d}\bar{\mathcal{H}}(\boldsymbol{\phi}, \boldsymbol{\varphi}, \mathbf{F})}{\mathrm{d}t} = \frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\phi}} \cdot \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} + \frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\varphi}} \cdot \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} + \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}} \cdot\cdot\, \dot{\mathbf{F}} \quad (30)$$

The preceding equation can be rewritten as

$$\frac{\mathrm{d}\bar{\mathcal{H}}(\boldsymbol{\phi}, \boldsymbol{\varphi}, \mathbf{F})}{\mathrm{d}t} = \left(\frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\varphi}} + \dot{\boldsymbol{\phi}}\right) \cdot \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} + \left(\frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\phi}} - \dot{\boldsymbol{\varphi}}\right) \cdot \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} + \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}} \cdot\cdot\, \dot{\mathbf{F}} \quad (31)$$

Recall the following relation:

$$\frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}} \cdot\cdot\, \dot{\mathbf{F}} = \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}} \cdot\cdot \nabla \dot{\boldsymbol{\varphi}} = \nabla \cdot \left(\dot{\boldsymbol{\varphi}} \cdot \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}}\right) - \left(\nabla \cdot \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}}\right) \cdot \dot{\boldsymbol{\varphi}} \quad (32)$$

Because of Gauss's theorem, the last term on the right-hand side of Eq. (32) follows the relationship

$$\int_{\mathcal{B}} \nabla \cdot \left(\dot{\boldsymbol{\varphi}} \cdot \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}}\right) \mathrm{d}V = \int_{\partial \mathcal{B}_\sigma} \dot{\boldsymbol{\varphi}} \cdot \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}} \cdot \hat{\mathbf{N}}\, \mathrm{d}A \quad (33)$$

In view of Eqs. (31–33), Eq. (29) leads to

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_{\mathcal{B}} \left(\left(\dot{\boldsymbol{\phi}} + \frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\varphi}} - \left(\nabla \cdot \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}}\right)\right) \cdot \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} + \left(\frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\phi}} - \dot{\boldsymbol{\varphi}}\right) \cdot \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t}\right) \mathrm{d}V - \int_{\partial \mathcal{B}_\sigma} \left(\bar{\mathbf{T}}(\mathbf{X}) - \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}} \cdot \hat{\mathbf{N}}\right) \cdot \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t}\, \mathrm{d}A = 0 \quad (34)$$

Within the Hamiltonian mechanics framework, canonical variables (namely, generalized coordinates and momentum) are linearly independent ([47], page 47 and [48], page 64). We would like to place an emphasis on the fact that the theorem of expended power contains governing equations and natural boundary conditions in the weighted residual form. Note that on the prescribed-displacement boundary surface, the differentials of canonical variables should vanish. Consequently, we readily observe that the rate of the autonomous Hamiltonian contains the strong form (the Hamiltonian version of Cauchy's equations of motion and the natural and essential boundary conditions), as follows:

$$\dot{\boldsymbol{\phi}} = -\frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\varphi}} + \left(\nabla \cdot \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}}\right), \qquad \dot{\boldsymbol{\varphi}} = \frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\phi}} \quad \text{in } \mathbf{X} \in \mathcal{B} \quad (35)$$

subject to the natural and essential boundary conditions

$$\bar{\mathbf{T}}(\mathbf{X}) = \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}} \cdot \hat{\mathbf{N}} \quad \text{on } \mathbf{X} \in \partial \mathcal{B}_\sigma, \qquad \boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} \quad \text{on } \mathbf{X} \in \partial \mathcal{B}_\varphi \quad (36)$$

Upon substitution of the Hamiltonian density function given in Eq. (26) into Eq. (35), we obtain

$$\dot{\boldsymbol{\phi}} = \rho_0 \mathbf{B} + \nabla \cdot \mathbf{P}, \qquad \dot{\boldsymbol{\varphi}} = \frac{\boldsymbol{\phi}}{\rho_0} \quad (37)$$

which is the Hamiltonian version of Cauchy's equations of motion for continuous-body dynamical systems.

It is noted that the rate of the autonomous Hamiltonian results in a hybrid weighted residual form. Equation (35) represents the scalar-termed governing equations of motion invariant to coordinate transformation within the Hamiltonian mechanics framework. The scalar here is the Hamiltonian density function. The dependent variables we want to solve for in the strong form are canonical variables (i.e., generalized coordinates and generalized momenta). The independent variables in the strong form consist of two variables (i.e., the time variable t and the spatial variable $\mathbf{X}$). In view of Eq. (35), the spatial dependent variable $\boldsymbol{\varphi}$ is of class $C^{2m}(\mathcal{B})$. Consequently, the exact solution for the governing equation is required to meet the continuity condition, as shown next:

$$\mathcal{C} = \left\{ \begin{array}{ll} \boldsymbol{\varphi}\colon \mathcal{B} \times \mathcal{T} \to \mathcal{S} \subset \mathbb{R}^3 | \boldsymbol{\varphi} \in C^{2m}(\mathcal{B}) & \\ \boldsymbol{\varphi} \in C^1(\mathcal{T}), m \geq 1, |\nabla \boldsymbol{\varphi}| > 0, & \boldsymbol{\varphi}|_{\partial \mathcal{B}_\varphi} = \bar{\boldsymbol{\varphi}} \end{array} \right\} \quad (38)$$

where $\bar{\boldsymbol{\varphi}}$ stands for the prescribed quantity on the prescribed-displacement boundary.

As shown in Eq. (34), the theorem of expended power contains the governing equations and boundary conditions in the Bubnov–Galerkin weighted residual form. Note that Eq. (34) includes the natural boundary conditions. As such, it is natural that we can derive a weak statement that allows a weaker continuity condition for trial and test functions as in traditional practices. In view of Eq. (31), Eq. (34) can yield the following weak form in the sense of the space:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_{\mathcal{B}} \dot{\boldsymbol{\phi}} \cdot \dot{\boldsymbol{\varphi}}\, \mathrm{d}V + \int_{\mathcal{B}} \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{F}} \cdot\cdot\, \dot{\mathbf{F}}\, \mathrm{d}V + \int_{\mathcal{B}} \frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\varphi}} \cdot \dot{\boldsymbol{\varphi}}\, \mathrm{d}V - \int_{\partial \mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}) \cdot \dot{\boldsymbol{\varphi}}\, \mathrm{d}A - \int_{\mathcal{B}} \left(\dot{\boldsymbol{\varphi}} - \frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\phi}}\right) \cdot \dot{\boldsymbol{\phi}}\, \mathrm{d}V = 0 \quad (39)$$

This is the so-called hybrid weak form; the dependent variables are the canonical variables: the canonical coordinates and momentum. It is true that the spatial discretization can be performed by use of Eq. (39). As the continuity condition becomes weakened, the displacement function can be of class $C^{m-1}(\mathcal{B})$. However, rather than employing the preceding equation to discretize the spatial domain, we propose an alternative approach for the spatial discretization in the subsequent section. Rather than using Eq. (39), we prefer to directly use Eq. (19), which is identical to the theorem of expended power; this is the key point in this paper. As an analogy to the functional in Ritz method for steady-state problems, the Hamiltonian for the continuous-body dynamical systems allows a weaker continuity condition for the admissible trial functions satisfying only essential boundary conditions as follows:

$$\mathcal{H}(\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\varphi}}) = \int_{\mathcal{B}} \bar{\mathcal{H}}(\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\varphi}}, \mathbf{F})\, \mathrm{d}V - \int_{\partial \mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X}) \cdot \tilde{\boldsymbol{\varphi}}(\mathbf{X}, t)\, \mathrm{d}A \quad (40)$$

where $\tilde{\boldsymbol{\phi}}$ and $\tilde{\boldsymbol{\varphi}}$ denote the trial functions satisfying the essential boundary conditions for the infinite-dimensional body. Note that the trial functions are not required to satisfy the natural boundary conditions.

IV. New Space-Discrete Hamiltonian Finite Element Formulation

The use of Eq. (34) is not suitable for discretizing the spatial domain, because it contains the strong form requiring a strong continuity condition. In contrast, Eq. (39) is acceptable to discretize the space, as it is a weak form allowing a weaker continuity condition for both the trial and test functions. However, it requires both the admissible trial and test functions satisfying the essential boundary conditions. Moreover, Eq. (39) requires, to some extent, a complicated derivation procedure. In view of these considerations, we provide new avenues and alternative viewpoints next.

In elastodynamics, a body is approximated by a finite number of subdomains, called elements [39,49] (say, finite elements) for the sake of illustration. Therefore, the whole domain is assumed to

consist of an assemblage of finite elements. It is supposed that the entire domain having a complex geometry can be broken up into subdomains. Then the admissible trial functions $\tilde{\boldsymbol{\varphi}}$ and $\tilde{\boldsymbol{\phi}}$ are required to be piecewise-continuous, and they should be continuous along the inter element boundaries. They may be approximated by a combination of interpolation functions and nodal variables as follows:

$$\begin{aligned}
\tilde{\boldsymbol{\varphi}}(\mathbf{X},t) &\approx \tilde{\boldsymbol{\varphi}}^h = \sum_{i=1}^{n_{\text{node}}} \mathbf{N}_i^h(\mathbf{X})\tilde{\boldsymbol{\varphi}}_i^h(t) = \mathbf{N}\mathbf{q} \\
\tilde{\boldsymbol{\phi}}(\mathbf{X},t) &\approx \tilde{\boldsymbol{\phi}}^h = \sum_{i=1}^{n_{\text{node}}} \mathbf{N}_i^h(\mathbf{X})\tilde{\boldsymbol{\phi}}_i^h(t) = \rho_0\mathbf{N}\dot{\mathbf{q}} \\
\mathbf{q}(t) &= \{\tilde{\boldsymbol{\varphi}}_1^h(t),\ldots,\tilde{\boldsymbol{\varphi}}_{n_{\text{node}}}^h(t)\}^T \in Q \subset \mathbb{R}^{n_{\text{dof}}} \\
\mathbf{p}(t) = \mathbf{M}\dot{\mathbf{q}}(t) \in T_q^*Q \subset \mathbb{R}^{n_{\text{dof}}}, &\qquad \mathbf{M} = \int_{\mathcal{B}} \rho_0\mathbf{N}^T\mathbf{N}\mathrm{d}V
\end{aligned} \tag{41}$$

where the trial functions $\tilde{\boldsymbol{\varphi}}^h(\mathbf{X},t)$ are of class $C^{m-1}(\mathcal{B})$ over the entire domain. The admissible trial functions belong to the subspace, denoted by $\mathcal{C}^h \subset \mathcal{C}$: that is,

$$\mathcal{C}^h = \left\{ \begin{array}{c} \tilde{\boldsymbol{\varphi}}^h(\mathbf{X},t)\colon \mathcal{B}\times\mathcal{T}\to \\ \mathbb{R}^3 | \tilde{\boldsymbol{\varphi}}^h \in W_2^m(\mathcal{B}) \subset C^{m-1}(\mathcal{B}), \qquad \mathbf{X}\in\mathcal{B} \\ \tilde{\boldsymbol{\varphi}}^h = \bar{\boldsymbol{\varphi}}, \qquad \text{on } \mathbf{X}\in\partial\mathcal{B}_\varphi \end{array} \right\} \tag{42}$$

Note that the essential boundary conditions become nodal position vectors on nodal points. Directly starting the discretization process via the rate of the Hamiltonian, we may choose the same admissible trial functions in Eq. (41) as the assumed displacement and canonical momentum field functions. Without involving test or weighting functions, a discrete Hamiltonian differential operator (which is discussed subsequently) is directly employed to obtain the semi-discrete first-order-in-time differential equations. For simplicity, we introduce a phase space with generalized coordinates and canonical momenta, denoted by

$$T^*Q = \bigcup_{q\in Q} T_q^*Q$$

that is,

$$T^*Q := \left\{ \begin{array}{c} (\mathbf{p},\mathbf{q}) | \mathbf{q}(t) = \{\tilde{\boldsymbol{\varphi}}_1^h(t),\ldots,\tilde{\boldsymbol{\varphi}}_{n_{\text{node}}}^h(t)\}^T \in Q \subset \mathbb{R}^{n_{\text{dof}}=3n_{\text{node}}} \\ \mathbf{p}(t) = \mathbf{M}\dot{\mathbf{q}}(t) \in T_q^*Q \subset \mathbb{R}^{n_{\text{dof}}=3n_{\text{node}}}, \quad \mathbf{M} = \int_{\mathcal{B}} \rho_0\mathbf{N}^T\mathbf{N}\,\mathrm{d}V \end{array} \right\} \tag{43}$$

Upon substitution of Eq. (41) into Eq. (29), the rate of the Hamiltonian can be discretized in space as follows:

$$\frac{\mathrm{d}\mathcal{H}^h(\tilde{\boldsymbol{\varphi}}^h,\tilde{\boldsymbol{\phi}}^h)}{\mathrm{d}t} = 0 \tag{44}$$

where the space-discrete autonomous Hamiltonian for the finite-dimensional systems can be simply constructed using the Hamiltonian density function as follows:

$$\begin{aligned}
\mathcal{H}^h(\tilde{\boldsymbol{\phi}}^h,\tilde{\boldsymbol{\varphi}}^h) &= \int_{\mathcal{B}} \bar{\mathcal{H}}^h(\tilde{\boldsymbol{\phi}}^h,\tilde{\boldsymbol{\varphi}}^h,\mathbf{F}^h)\,\mathrm{d}V - \int_{\partial\mathcal{B}_\sigma} \bar{\mathbf{T}}(\mathbf{X})\cdot\tilde{\boldsymbol{\varphi}}^h(\mathbf{X},t)\,\mathrm{d}A \\
&= \mathcal{H}^h(\mathbf{p},\mathbf{q})
\end{aligned} \tag{45}$$

As shown in the preceding equation, the Hamiltonian can be discretized by the admissible trial functions satisfying the essential boundary conditions. The trial function $\tilde{\boldsymbol{\varphi}}^h \in W_2^m(\mathcal{B}) = H^m(\mathcal{B})$ is of class $C^{m-1}(\mathcal{B})$, whereas another trial function $\tilde{\boldsymbol{\phi}}^h \in W_2^0 = H^0(\mathcal{B})$ is of class $C^0(\mathcal{B})$. As shown in Eq. (44), the Hamiltonian is not a function with convexity. We do not minimize the Hamiltonian to get some approximate solution (the Ritz method for steady-state problems requires a functional to be minimized). Instead of minimizing the Hamiltonian, we simply have a differential operator to obtain semidiscrete finite element equations of motion. In terms of the generalized coordinates and canonical momenta, Eq. (44) can be rewritten as

$$\frac{\mathrm{d}\mathcal{H}^h(\mathbf{p},\mathbf{q})}{\mathrm{d}t} = \left(\dot{\mathbf{p}} + \frac{\partial\mathcal{H}^h}{\partial\mathbf{q}}\right)\cdot\dot{\mathbf{q}} - \left(\dot{\mathbf{q}} - \frac{\partial\mathcal{H}^h}{\partial\mathbf{p}}\right)\cdot\dot{\mathbf{p}} = 0 \tag{46}$$

where the space-discrete autonomous Hamiltonian consisting of the space-discrete autonomous kinetic and potential energies is constructed as follows:

$$\begin{aligned}
\mathcal{H}^h(\mathbf{p},\mathbf{q}) &= \frac{1}{2}\mathbf{p}\cdot\mathbf{M}^{-1}\mathbf{p} + \mathcal{U}_{\text{int}}^h - \mathbf{q}\cdot\mathbf{F}_{\text{ext}} \\
\mathcal{U}_{\text{int}}^h = \int_{\mathcal{B}} \rho_0\Psi(\mathbf{F}^h)\,\mathrm{d}V, &\qquad \mathbf{M} = \int_{\mathcal{B}} \rho_0\mathbf{N}^T\mathbf{N}\,\mathrm{d}V \\
\frac{\partial\mathcal{H}^h(\mathbf{p},\mathbf{q})}{\partial\mathbf{q}} = \mathbf{F}_{\text{int}} - \mathbf{F}_{\text{ext}}, &\qquad \mathbf{F}_{\text{int}} = \int_{\mathcal{B}} \bar{\mathbf{B}}^T\mathbf{P}\,\mathrm{d}V \\
\mathbf{F}_{\text{ext}} &= \int_{\mathcal{B}} \rho_0\mathbf{N}^T\mathbf{B}\,\mathrm{d}V + \int_{\partial\mathcal{B}_\sigma} \mathbf{N}^T\bar{\mathbf{T}}\,\mathrm{d}A
\end{aligned} \tag{47}$$

where $\bar{\mathbf{B}}$ denotes the strain matrix and N is the interpolation matrix. The preceding expression holds regardless of whether material/geometric nonlinearity of the body exists or not. Note that the admissible trial functions $\tilde{\boldsymbol{\varphi}}^h(\mathbf{X},t)$ we used previously to discretize the rate of the Hamiltonian are of class $C^{m-1}(\mathcal{B})$. This continuity condition is identical to that of the weak form given in Eq. (39). As discussed in the preceding, the theorem of expended power contains both governing equations and natural boundary conditions just as the weak statement given in Eq. (39) does. Hence, emphasis is placed on the fact that the discretization of the rate of the Hamiltonian shown in Eq. (46) becomes identical to the discretization of the weak form given in Eq. (39). Therefore, the rate of the Hamiltonian has a mechanism by which we can make the weighted residual average vanish, similar to that obtained when it is discretized by a suitable trial function. Note that the rate of Hamiltonian does not require governing equations and boundary conditions to be given, but both governing equations and boundary conditions are required to be given to derive the weak form given in Eq. (39).

In contrast to traditional practices, for discretizing in space, we simply adopt the following procedure. In view of Eq. (46), we introduce the space-discrete Hamiltonian differential operators $\mathfrak{H}_{D1}$ and $\mathfrak{H}_{D2}$, by which the space-discrete finite element equation of motion can be readily obtained after constructing the space-discrete Hamiltonian, which only entails the use of trial functions. Thus, upon substituting the space-discrete Hamiltonian into the space-discrete Hamiltonian differential operators, we have the following:

$$\begin{aligned}
\mathfrak{H}_{D1}\colon \dot{\mathbf{p}} + \frac{\partial}{\partial\mathbf{q}}, &\qquad \mathfrak{H}_{D2}\colon \dot{\mathbf{q}} - \frac{\partial}{\partial\mathbf{p}} \\
\mathfrak{H}_{D1}(\mathcal{H}^h) = \dot{\mathbf{p}} + \frac{\partial\mathcal{H}^h}{\partial\mathbf{q}} = \mathbf{0}, &\qquad \mathfrak{H}_{D2}(\mathcal{H}^h) = \dot{\mathbf{q}} - \frac{\partial\mathcal{H}^h}{\partial\mathbf{p}} = \mathbf{0}
\end{aligned} \tag{48}$$

where the dependent state variable is the space-discrete scalar function (namely, the space-discrete autonomous Hamiltonian). *Note that the preceding discrete Hamiltonian differential operators imply that errors associated with governing equations and natural boundary conditions are implicitly minimized when the admissible trial functions satisfying the essential boundary conditions are chosen*. Thus, we readily have the space-discrete finite element equation of motion governing the mechanical dynamical behavior of the continuous body within the Hamiltonian mechanics framework as follows,

$$\dot{\mathbf{p}} + \mathbf{F}_{\text{int}} - \mathbf{F}_{\text{ext}} = \mathbf{0} \tag{49}$$

which is subject to a subsidiary velocity condition:

$$\dot{\mathbf{q}} - \mathbf{M}^{-1}\mathbf{p} = \mathbf{0} \tag{50}$$

As an analogy, in contrast to elastostatics, the discrete Hamiltonian differential operator for the finite element equilibrium equations can be readily reduced to

$$\mathfrak{S}_S = \frac{\partial}{\partial \mathbf{q}}, \qquad \mathfrak{S}_S(\mathcal{H}^h) = \frac{\partial \mathcal{H}^h}{\partial \mathbf{q}} = \mathbf{F}_{\text{int}} - \mathbf{F}_{\text{ext}} = \mathbf{0} \tag{51}$$

Equation (51) is now the familiar representation with the generalized coordinates corresponding to the Ritz coefficients. Note that the preceding discrete Hamiltonian differential operator implies that equilibrium equations and natural boundary conditions are implicitly minimized when the admissible trial functions are chosen to satisfy the essential boundary conditions. Hence, the space-discrete autonomous Hamiltonian in Eq. (51) is equivalent to the spatially discretized total potential energy in the principle of stationary total potential energy. Taking the time integral over the given time interval $t \in \mathcal{T} = [t_1, t_2]$ for both sides in Eq. (46), we have the statement of the weighted residual in time given by

$$\begin{aligned}&\mathcal{H}^h(\mathbf{p}, \mathbf{q})|_{t=t_2} - \mathcal{H}^h(\mathbf{p}, \mathbf{q})|_{t=t_1} \\ &\quad = \int_{t_1}^{t_2} \{(\dot{\mathbf{p}} + \mathbf{F}_{\text{int}} - \mathbf{F}_{\text{ext}}) \cdot \dot{\mathbf{q}} - (\dot{\mathbf{q}} - \mathbf{M}^{-1}\mathbf{p}) \cdot \dot{\mathbf{p}}\}\, \mathrm{d}t = 0\end{aligned} \tag{52}$$

The preceding equation may be regarded as an unconstrained formulation with two dependent field variables in the time domain; thus, we have the hybrid two-field problem within the Hamiltonian mechanics framework. Note that Eq. (52) in the Hamiltonian mechanics framework represents the hybrid weighted residual form in time. Hence, Eq. (52) plays an important role of facilitating the discretization in the time domain. In general, the time-weighted residual method [43] or the time-finite-element procedure [50] can be applied to Eq. (52) to derive time-stepping schemes. Note that Eq. (52) also takes the hybrid weighted residual form in time. The weighting function is the generalized velocity. Hence, Eq. (52) can be used as a critical equation to discretize the time domain.

V. Illustrative Examples

For illustration, we present the finite element formulations proposed in this exposition for two different types of structural finite elements, such as the Euler–Bernoulli beam and the Reissner–Mindlin plate. It is assumed that the continuous-dynamical systems we consider here are conservative and that the materials undergo perfect elastic deformation without producing entropy.

A. Euler–Bernoulli Beam

Let us consider the dynamical behavior of the Euler–Bernoulli beam as a simple numerical example that is typical of the class of continuous-dynamical systems. It is assumed that the normal plane perpendicular to the axial axis of the beam remains in the plane normal to the axial axis after deformation and that the beam is not subject to the axial force. Thus, the strain energy is due to only the bending moment $M_B(x, t)$ of the beam. Then the dynamical behavior of the Euler–Bernoulli beam is governed by one differential equation, because of the presence of one dependent variable: namely, the transverse deflection $w(x, t)$. Its derivative is the angle of rotation $\varphi(x, t)$ due to bending moment acting on the beam. The total rotation of the Euler–Bernoulli beam is defined as

$$\frac{\partial w(x,t)}{\partial x} = w'(x,t) = \varphi(x,t) \tag{53}$$

It means that the deformation of the beam is associated with rectilinear and rotational motions at the same time. Therefore, the kinetic energy consists of the transverse translational kinetic energies and the rotational kinetic energies such as

$$\begin{aligned}&\mathcal{K}(\dot{w}) = \frac{1}{2}\int_l \dot{\mathbf{u}} \cdot \boldsymbol{\rho}\, \dot{\mathbf{u}}\, \mathrm{d}x, \qquad \boldsymbol{\rho} = \begin{bmatrix} \rho(x)A(x) & 0 \\ 0 & \rho(x)I(x) \end{bmatrix} \\ &\mathbf{u}(x,t) = [w(x,t), \varphi(x,t)]^T, \qquad \dot{\mathbf{u}}(x,t) = [\dot{w}(x,t), \dot{\varphi}(x,t)]^T\end{aligned} \tag{54}$$

where $I(x)$ denotes the sectional moment of area about the neutral axis per unit length about the center of the cross section, $\rho(x)$ density, and $A(x)$ the cross-sectional area. Further, the total potential energy of the beam subject to the distributed load $p(x)$ may be given by

$$\begin{aligned}\mathcal{U}(w) &= \frac{1}{2}\int_l M_B(x,t)\varphi'\, \mathrm{d}x - \int_l p(x)w(x,t)\, \mathrm{d}x \\ &= \frac{1}{2}\int_l M_B(x,t)w''\, \mathrm{d}x - \int_l p(x)w(x,t)\, \mathrm{d}x \\ w'' &= \frac{\partial^2 w(x,t)}{\partial x^2}, \qquad \varphi' = \frac{\partial \varphi(x,t)}{\partial x}\end{aligned} \tag{55}$$

where the bending moment $M_B(x, t)$ may be given as

$$M_B(x,t) = EI(x)\varphi'(x,t) = EI(x)w''(x,t) \tag{56}$$

Thus, the total potential energy may be rewritten as

$$\begin{aligned}&\mathcal{U}(w) = \frac{1}{2}\int_l (\mathbf{u}' \cdot \mathbf{D}_b \mathbf{u}')\, \mathrm{d}x - \int_l p(x)\mathbf{a} \cdot \mathbf{u}\, \mathrm{d}x, \qquad \boldsymbol{a} = [1 \quad 0]^T \\ &\mathbf{D}_b = \begin{bmatrix} 0 & 0 \\ 0 & EI(x) \end{bmatrix}, \qquad \mathbf{u}'(x,t) = \{w'(x,t), \varphi'(x,t)\}^T\end{aligned} \tag{57}$$

The continuous Lagrangian of the Euler–Bernoulli beam can be found as

$$\mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) = \int_l \bar{\mathcal{L}}(\dot{\mathbf{u}}, \mathbf{u}')\, \mathrm{d}x + \int_l p(x)\mathbf{a} \cdot \mathbf{u}\, \mathrm{d}x \tag{58}$$

where the Lagrangian density function is defined as

$$\bar{\mathcal{L}}(\dot{\mathbf{u}}, \mathbf{u}') = \tfrac{1}{2}\dot{\mathbf{u}} \cdot \boldsymbol{\rho}\, \dot{\mathbf{u}} - \tfrac{1}{2}\mathbf{u}' \cdot \mathbf{D}_b \mathbf{u}' \tag{59}$$

In view of Eq. (59), the generalized momenta $\boldsymbol{\phi}$ can be given by

$$\boldsymbol{\phi} = \frac{\partial \bar{\mathcal{L}}(\dot{\mathbf{u}}, \mathbf{u}')}{\partial \dot{\mathbf{u}}}, \qquad \boldsymbol{\phi} = [\phi \quad \phi_1]^T \tag{60}$$

Note that $\dot{w}'$ is dependent on $\dot{w}$. In the Euler–Bernoulli beam theory, the dependent variable is the transverse deflection $w(x, t)$ (in other words, it is the single-field problem). Thus, we say that the notations of the generalized momenta in Eq. (60) are introduced for convenience.

Then the theorem of expended power gives

$$\frac{\mathrm{d}\mathcal{H}(\mathbf{u}, \boldsymbol{\phi})}{\mathrm{d}t} = 0 \tag{61}$$

where the autonomous Hamiltonian of the Euler–Bernoulli beam is given by

$$\mathcal{H}(\mathbf{u}, \boldsymbol{\phi}) = \int_l \bar{\mathcal{H}}(\boldsymbol{\phi}\mathbf{u}')\, \mathrm{d}x - \int_l p(x)\mathbf{a} \cdot \mathbf{u}\, \mathrm{d}x \tag{62}$$

and the Hamiltonian density function is defined as

$$\bar{\mathcal{H}}(\mathbf{u}, \boldsymbol{\phi}, \mathbf{u}') = \tfrac{1}{2}\boldsymbol{\phi} \cdot \boldsymbol{\rho}^{-1}\boldsymbol{\phi} + \tfrac{1}{2}\mathbf{u}' \cdot \mathbf{D}_b \mathbf{u}' \tag{63}$$

Note that Eq. (61) for scleronomic systems can be rewritten as

$$\frac{\mathrm{d}\mathcal{H}(\mathbf{u}, \boldsymbol{\phi})}{\mathrm{d}t} = \int_l \frac{\mathrm{d}\bar{\mathcal{H}}(\mathbf{u}, \boldsymbol{\phi}\mathbf{u}')}{\mathrm{d}t}\, \mathrm{d}x - \int_l p(x)\mathbf{a} \cdot \dot{\mathbf{u}}\, \mathrm{d}x = 0 \tag{64}$$

By use of the chain rule, the rate of the Hamiltonian density function gives

$$\frac{\mathrm{d}\bar{\mathcal{H}}(\mathbf{u}, \boldsymbol{\phi}\mathbf{u}')}{\mathrm{d}t} = \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{u}} \cdot \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\phi}} \cdot \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} + \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{u}'} \cdot \dot{\mathbf{u}}' \tag{65}$$

The preceding equation can be rewritten as

$$\frac{d\bar{\mathcal{H}}(\mathbf{u}, \boldsymbol{\phi}\mathbf{u}')}{dt} = \left(\frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{u}} + \dot{\boldsymbol{\phi}}\right) \cdot \frac{d\mathbf{u}}{dt} + \left(\frac{\partial \bar{\mathcal{H}}}{\partial \boldsymbol{\phi}} - \dot{\mathbf{u}}\right) \cdot \frac{d\boldsymbol{\phi}}{dt} + \frac{\partial \bar{\mathcal{H}}}{\partial \mathbf{u}'} \cdot \dot{\mathbf{u}}' \tag{66}$$

The first term on the right-hand side of Eq. (65) can be given by

$$\left(\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}}+\dot{\boldsymbol{\phi}}\right)\cdot\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}=\left(\frac{\partial\bar{\mathcal{H}}}{\partial w}+\dot{\phi}\right)\dot{w}+\left(\frac{\partial\bar{\mathcal{H}}}{\partial w'}+\dot{\phi}_1\right)\dot{w}'$$
$$=\left(\frac{\partial\bar{\mathcal{H}}}{\partial w}-\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w'}+\dot{\phi}_1\right)+\dot{\phi}\right)\dot{w}+\frac{\partial}{\partial x}\left(\left(\frac{\partial\bar{\mathcal{H}}}{\partial w'}+\dot{\phi}_1\right)\dot{w}\right) \quad (67)$$

The second term on the right-hand side of Eq. (65) can be given by

$$\left(\frac{\partial\bar{\mathcal{H}}}{\partial\boldsymbol{\phi}}-\dot{\mathbf{u}}\right)\cdot\frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t}=\left(\frac{\partial\bar{\mathcal{H}}}{\partial\phi}-\dot{w}\right)\dot{\phi}+\left(\frac{\partial\bar{\mathcal{H}}}{\partial\phi_1}-\dot{w}'\right)\dot{\phi}_1=\left(\frac{\partial\bar{\mathcal{H}}}{\partial\phi}-\dot{w}\right)\dot{\phi} \quad (68)$$

Note that $\dot{w}'=\partial\bar{\mathcal{H}}/-\partial\varphi_1$. The last term on the right-hand side of Eq. (52) can be given by

$$\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}'}\cdot\dot{\mathbf{u}}'=\frac{\partial}{\partial x}\left\{\left(\frac{\partial\bar{\mathcal{H}}}{\partial w'}-\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w''}\right)\right)\dot{w}+\frac{\partial\bar{\mathcal{H}}}{\partial w''}\dot{w}'\right\}$$
$$+\left\{\frac{\partial^2}{\partial x^2}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\dot{w}'}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w'}\right)\right\}\dot{w} \quad (69)$$

Hence, in view of Eqs. (67–69), the rate of the Hamiltonian leads to

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}=\int_l\left[\left(\frac{\partial\bar{\mathcal{H}}}{\partial w}-\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w'}\right)-\frac{\partial}{\partial x}(\dot{\phi}_1)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\dot{w}'}\right)\right.\right.$$
$$\left.-p(x)+\dot{\phi}\right)\dot{w}+\left(\frac{\partial\bar{\mathcal{H}}}{\partial\phi}-\dot{w}\right)\dot{\phi}\Big]\,\mathrm{d}x$$
$$+\left[\left(\dot{\phi}_1-\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w''}\right)\right)\dot{w}\right]_0^l+\left[\frac{\partial\bar{\mathcal{H}}}{\partial w''}\dot{w}'\right]_0^l=0 \quad (70)$$

where both governing equations and natural boundary conditions are included. Consequently, we have derived the governing equation of motion with the traction boundary conditions via the rate of the Hamiltonian involving the Hamiltonian density function. Because the differentials of the dependent variables are linearly independent, the governing equation of the Euler–Bernoulli beam in terms of the Hamiltonian density function can be obtained as

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w'}\right)-\frac{\partial^2}{\partial x^2}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w''}\right)+\frac{\partial}{\partial x}\left(\frac{\mathrm{d}\phi_1}{\mathrm{d}t}\right)-\frac{\partial\bar{\mathcal{H}}}{\partial w}+p(x)$$
$$\dot{w}=\frac{\partial\bar{\mathcal{H}}}{\partial\phi} \quad (71)$$

subject to the boundary conditions

$$\left(\dot{\phi}_1-\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w''}\right)\right)\dot{w}\Big|_0^l=0,\qquad\frac{\partial\bar{\mathcal{H}}}{\partial w''}\dot{w}'\Big|_0^l=0 \quad (72)$$

Subsequently, upon substitution of the Hamiltonian density function in Eq. (63) into Eqs. (71) and (72), we obtain the differential equations of motion as follows:

$$\dot{\phi}=\frac{\partial}{\partial x}(\rho(x)I(x)\ddot{w}')-\frac{\partial^2}{\partial x^2}(EI(x)w'')+p(x)=0$$
$$\dot{w}=\frac{\phi}{\rho(x)A(x)} \quad (73)$$

subject to the boundary conditions

$$EI(x)w''\dot{w}'|_0^l=0,\qquad\left(\frac{\partial}{\partial x}(EI(x)w'')-\rho I(x)\ddot{w}'\right)\dot{w}|_0^l=0 \quad (74)$$

We note that the exact solution of the transverse deflection should satisfy the continuity condition [viz., $w(x,t)\in C^4(l)$]. However, as discussed in the previous section, the autonomous Hamiltonian of the Euler–Bernoulli beam given in Eq. (62) allows trial functions satisfying a weaker continuity condition. Note that the trial function $w^h(x,t)$ is of class $H^2(l)\subset C^1$. This is a weaker continuity condition, compared with the condition for Eq. (73). Thus, the Euler–Bernoulli beam element is often called a C^1 element.

Next, by making use of only the trial function, we can develop a set of first-order semidiscrete finite element equations of motion via the space-discrete Hamiltonian formulation, rather than using the weak form that is derived from the weighted residual form. Consider one element with two nodes. The trial displacement function may be approximated using the Hermite polynomial interpolation functions as follows:

$$w^h(x,t)=\mathbf{N}(x)\cdot\mathbf{q}(t),\qquad\frac{\partial w^h(x,t)}{\partial x}=\mathbf{B}_1(x)\cdot\mathbf{q}(t)$$
$$\mathbf{B}_1(x)=\frac{\partial\mathbf{N}(x)}{\partial x}\qquad\mathbf{q}(t)=\left(w_1^h(t),\frac{\partial w_1^h}{\partial x}(t),w_2^h(t),\frac{\partial w_2^h}{\partial x}(t)\right)$$
$$\dot{\mathbf{q}}(t)=\left(\dot{w}_1^h(t),\frac{\partial\dot{w}_1^h}{\partial x}(t),\dot{w}_2^h(t),\frac{\partial\dot{w}_2^h}{\partial x}(t)\right) \quad (75)$$

As the space-discrete kinetic energy, we obtain

$$\mathcal{K}^h(\mathbf{p})=\tfrac{1}{2}\mathbf{p}(t)\cdot\mathbf{M}^{-1}\mathbf{p}(t),\qquad\mathbf{p}(t)=\mathbf{M}\dot{\mathbf{q}}(t) \quad (76)$$

where the consistent mass matrix is given by

$$\mathbf{M}=\mathbf{M}_A+\mathbf{M}_I,\qquad\mathbf{M}_A=\int_l\rho A(x)\mathbf{N}^T\mathbf{N}\,\mathrm{d}x$$
$$\mathbf{M}_I=\int_l\rho I(x)\mathbf{B}_1^T\mathbf{B}_1\,\mathrm{d}x \quad (77)$$

In addition, the space-discrete potential energy can be obtained as

$$\mathcal{U}^h(\mathbf{q})=\tfrac{1}{2}\mathbf{q}(t)\cdot\mathbf{K}\mathbf{q}(t)-\mathbf{q}(t)\cdot\mathbf{F}_{\text{ext}} \quad (78)$$

where the stiffness matrix and the external force can be given by

$$\mathbf{K}=\int_l EI(x)\mathbf{B}_2^T\mathbf{B}_2\,\mathrm{d}x,\qquad\mathbf{B}_2(x)=\frac{\partial^2\mathbf{N}(x)}{\partial x^2}$$
$$\mathbf{F}_{\text{ext}}=\int_l p(x)\mathbf{N}^T(x)\,\mathrm{d}x \quad (79)$$

We now construct the space-discrete autonomous Hamiltonian:

$$\mathcal{H}^h(\mathbf{p},\mathbf{q})=\mathcal{K}^h(\mathbf{p})+\mathcal{U}^h(\mathbf{q})=\tfrac{1}{2}\mathbf{p}(t)\cdot\mathbf{M}^{-1}\mathbf{p}(t)$$
$$+\tfrac{1}{2}\mathbf{q}(t)\cdot\mathbf{K}\mathbf{q}(t)-\mathbf{q}(t)\cdot\mathbf{F}_{\text{ext}} \quad (80)$$

Substituting the space-discrete autonomous Hamiltonian given in Eq. (80) into the discrete Hamiltonian differential operators in Eq. (48) readily yields the semidiscrete equations of motion as

$$\mathfrak{S}_{D1}(\mathcal{H}^h)=\dot{\mathbf{p}}+\mathbf{K}\mathbf{q}-\mathbf{F}_{\text{ext}}=0,\quad\mathfrak{S}_{D2}(\mathcal{H}^h)=\dot{\mathbf{q}}-\mathbf{M}^{-1}\mathbf{p}=0 \quad (81)$$

As a numerical implementation for the Euler–Bernoulli beam, we conduct the transient dynamical analysis of a cantilever Euler–Bernoulli beam. As shown in Fig. 1, material properties, initial configuration, and loading conditions are depicted. The exact solution is regarded as the solution we obtain when the number of elements equals 50. Thus, Fig. 1 shows the tip deflection with 50 elements and the time-step size of 0.001 s. To demonstrate monotonic convergence, we show successive mesh refinements. A uniform h refinement on the mesh size is carried out to describe a convergence graph, as shown in Fig. 1.

B. Reissner–Mindlin Plate

Next, we shall discuss the Reissner–Mindlin plate of uniform thickness h as a two-dimensional problem for illustration. We assume that the transverse normal stress of the moderately thick plate is negligible in plane stress, compared with the other stresses components, and that the plate is uniformly subject to transverse distributed load $p(x,y)$ per unit area of the midsurface in the z direction. Furthermore, it is assumed that there are no body forces

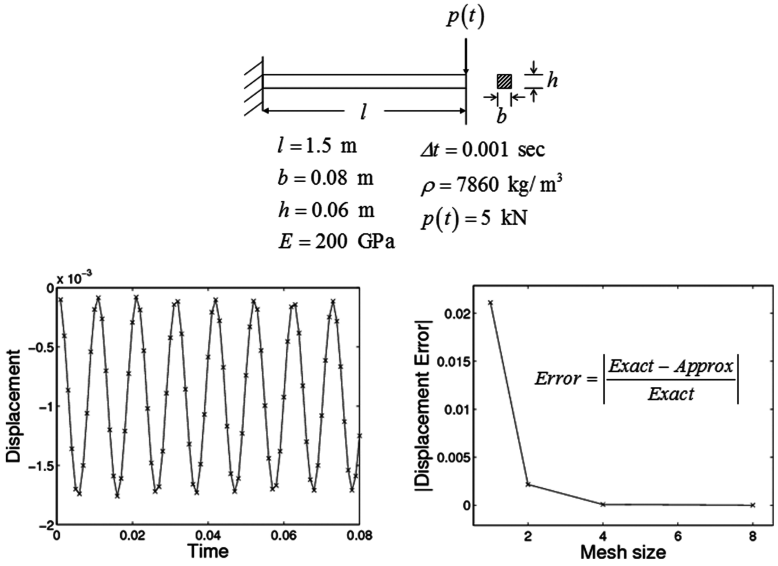

Fig. 1 Transient dynamical response of a cantilever beam.

acting in the plate and no shear forces on the flat surfaces. For the sake of brevity, the membrane behavior of the plate is not considered; the deformation is due to the bending and shear. In contrast to the Kirchhoff–Love thin plate, the transverse shear strains due to the moderate thickness effect are taken into account as an extension of the Timoshenko beam to the plate. Then the dynamical behavior of the Reissner–Mindlin plate is governed by three differential governing equations because of the presence of three dependent variables: namely, the transverse deflection $w(x, y, t)$ and two rotations $\theta_x(x, y, t)$ and $\theta_y(x, y, t)$. The reason for us to illustrate the Reissner–Mindlin plate is that the plate reflects the rotary inertia effect as well as the translational kinetic motion effect and the strain energy consists of the bending strain and shear strain energies. Being confined to the small-displacement theory, the displacements corresponding to such deformations at any point of the plate are given by

$$u(x, y, z, t) = z\theta_x, \qquad v(x, y, z, t) = -z\theta_y \tag{82}$$

which is often called the first-order shear deformation theory of the plate. Thus, the normal and shear strains are obtained as

$$\varepsilon_x = z\frac{\partial \theta_x}{\partial x}, \qquad \varepsilon_y = -z\frac{\partial \theta_y}{\partial y}, \qquad \gamma_{xy} = z\left(\frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x}\right)$$
$$\gamma_{xz} = \frac{\partial w(x, y, t)}{\partial x} + \theta_x, \qquad \gamma_{yz} = \frac{\partial w(x, y, t)}{\partial y} - \theta_y \tag{83}$$

and the in-plane stress components are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{zE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ -\frac{\partial \theta_y}{\partial y} \\ \left(\frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x}\right) \end{Bmatrix} \tag{84}$$

The transverse shear stress components are given by

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = G\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = G\begin{Bmatrix} \frac{\partial w(x,y,t)}{\partial x} + \theta_x \\ \frac{\partial w(x,y,t)}{\partial y} - \theta_y \end{Bmatrix} \tag{85}$$

In favor of matrix and vector notation, the strain energy is composed of the bending strain energy $\mathcal{U}_b(\kappa)$ and the shear strain energy $\mathcal{U}_s(\gamma)$, as shown next:

$$\mathcal{U}_b = \frac{1}{2}\iint_\Omega \boldsymbol{\kappa}\cdot\mathbf{D}_b\boldsymbol{\kappa}\,\mathrm{d}x\,\mathrm{d}y, \qquad \mathcal{U}_s = \frac{1}{2}\iint_\Omega \boldsymbol{\gamma}\cdot\mathbf{G}_s\boldsymbol{\gamma}\,\mathrm{d}x\,\mathrm{d}y$$
$$\boldsymbol{\kappa}^T = \begin{bmatrix} \frac{\partial \theta_x}{\partial x} & -\frac{\partial \theta_y}{\partial y} & \left(\frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x}\right) \end{bmatrix}, \qquad \gamma^T = \begin{bmatrix} \frac{\partial w}{\partial x} + \theta_x & \frac{\partial w}{\partial y} - \theta_y \end{bmatrix}$$
$$D = \frac{Eh^3}{12(1-\nu^2)}, \qquad \mathbf{D}_b = D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$
$$G_s = \begin{bmatrix} Ghk & 0 \\ 0 & Ghk \end{bmatrix} \tag{86}$$

where k denotes the shear correction factor, G is the shear modulus, D is the plate flexural rigidity, and Ω is the domain of interest in the midsurface of the flat plate. In the dynamical behavior of the plate, the deformation of the plate is associated with transverse and rotational motions at the same time. Therefore, the kinetic energy $\mathcal{K}(\dot{\mathbf{u}})$ consists of the transverse translational kinetic energies and the rotational kinetic energies such as

$$\mathcal{K}(\dot{\mathbf{u}}) = \frac{1}{2}\int_\Omega \dot{\mathbf{u}}\cdot\boldsymbol{\rho}\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}y, \qquad \boldsymbol{\rho} = \begin{bmatrix} \rho h & 0 & 0 \\ 0 & \rho I_p & 0 \\ 0 & 0 & \rho I_p \end{bmatrix}$$
$$\dot{\mathbf{u}} = \begin{Bmatrix} \dot{w} \\ \dot{\theta}_x \\ \dot{\theta}_y \end{Bmatrix}, \qquad I_p = \frac{h^3}{12} \tag{87}$$

We have the continuous Lagrangian for the Reissner–Mindlin plate as follows:

$$\mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) = \int_\Omega \bar{\mathcal{L}}\mathrm{d}x\mathrm{d}y + \int_\Omega p(x, y)\mathbf{a}\cdot\mathbf{u}\mathrm{d}x\mathrm{d}y$$
$$\mathbf{u} = (w, \theta_x, \theta_y), \qquad \dot{\mathbf{u}} = (\dot{w}, \dot{\theta}_x, \dot{\theta}_y), \qquad \mathbf{a} = (1, 0, 0)$$
$$\mathbf{u}_{,x} = \left(\frac{\partial w}{\partial x}, \frac{\partial \theta_x}{\partial x}, \frac{\partial \theta_y}{\partial x}\right), \qquad \mathbf{u}_{,y} = \left(\frac{\partial w}{\partial y}, \frac{\partial \theta_x}{\partial y}, \frac{\partial \theta_y}{\partial y}\right) \tag{88}$$

where the Lagrangian density function can be defined as

$$\bar{\mathcal{L}}(\mathbf{u},\dot{\mathbf{u}},\mathbf{u}_{,x},\mathbf{u}_{,y}) = \tfrac{1}{2}\dot{\mathbf{u}}\cdot\boldsymbol{\rho}\,\dot{\mathbf{u}} - \tfrac{1}{2}\boldsymbol{\kappa}\cdot\mathbf{D}_b\boldsymbol{\kappa} - \tfrac{1}{2}\boldsymbol{\gamma}\cdot\mathbf{G}_s\boldsymbol{\gamma} \tag{89}$$

Therefore, the components of the canonical momenta

$$\boldsymbol{\phi}\,(x,t) = (\phi_1(x,y,t),\phi_2(x,y,t),\phi_2(x,y,t))$$

are found as

$$\boldsymbol{\phi} = \frac{\partial\bar{\mathcal{L}}(\mathbf{u},\dot{\mathbf{u}},\mathbf{u}_{,x},\mathbf{u}_{,y})}{\partial\dot{\mathbf{u}}} \tag{90}$$

The Hamiltonian density function can be given via the Legendre transformation by

$$\bar{\mathcal{H}}(\mathbf{u},\boldsymbol{\phi},\mathbf{u}_{,x},\mathbf{u}_{,y}) = \boldsymbol{\phi}(x,y,t)\cdot\dot{\mathbf{u}}(x,y,t) - \bar{\mathcal{L}}(\mathbf{u},\dot{\mathbf{u}},\mathbf{u}_{,x},\mathbf{u}_{,y}) \tag{91}$$

Then the Hamiltonian density function can be given by

$$\begin{gathered}\bar{\mathcal{H}} = \tfrac{1}{2}\boldsymbol{\phi}\cdot\boldsymbol{\rho}^{-1}\boldsymbol{\phi} + \tfrac{1}{2}\boldsymbol{\kappa}\cdot\boldsymbol{D}_b\boldsymbol{\kappa} + \tfrac{1}{2}\boldsymbol{\gamma}\cdot\mathbf{G}_s\\ \boldsymbol{\gamma} = \tfrac{1}{2}\boldsymbol{\phi}\cdot\boldsymbol{\rho}^{-1}\boldsymbol{\phi} + \tfrac{1}{2}\boldsymbol{\kappa}\cdot\boldsymbol{M}_b + \tfrac{1}{2}\boldsymbol{\gamma}\cdot\boldsymbol{Q}_s, \qquad \boldsymbol{\phi} = \boldsymbol{\rho}\dot{\boldsymbol{u}}\\ \boldsymbol{M}_b = \boldsymbol{D}_b\boldsymbol{\kappa}, \qquad \boldsymbol{Q}_s = \boldsymbol{G}_s\boldsymbol{\gamma}\end{gathered} \tag{92}$$

where the generalized momentum and the stress resultants are given by

$$\begin{gathered}\boldsymbol{p}^T = [\,\rho\dot{w}h \quad \rho\dot{\theta}_x I_p \quad \rho\dot{\theta}_y I_p\,], \qquad \boldsymbol{M}_b^T = [\,M_{xx} \quad M_{yy} \quad M_{xy}\,]\\ \boldsymbol{Q}_s^T = [\,Q_x \quad Q_y\,]\end{gathered} \tag{93}$$

Note that the rate of the Hamiltonian for scleronomic systems can be rewritten as

$$\frac{\mathrm{d}\mathcal{H}(\mathbf{u},\boldsymbol{\phi})}{\mathrm{d}t} = \int_\Omega \frac{\mathrm{d}\bar{\mathcal{H}}}{\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega p(x,y)\mathbf{a}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{94}$$

By use of the chain rule, the rate of the Hamiltonian density function gives

$$\frac{\mathrm{d}\bar{\mathcal{H}}(\mathbf{u},\boldsymbol{\phi},\mathbf{u}_{,x},\mathbf{u}_{,y})}{\mathrm{d}t} = \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{\partial\bar{\mathcal{H}}}{\partial\boldsymbol{\phi}}\cdot\dot{\boldsymbol{\phi}} + \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\cdot\dot{\mathbf{u}}_{,x} + \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\cdot\dot{\mathbf{u}}_{,y} \tag{95}$$

The preceding equation can be rewritten as

$$\frac{\mathrm{d}\bar{\mathcal{H}}}{\mathrm{d}t} = \left(\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}} + \dot{\boldsymbol{\phi}}\right)\cdot\dot{\mathbf{u}} + \left(\frac{\partial\bar{\mathcal{H}}}{\partial\boldsymbol{\phi}} - \dot{\mathbf{u}}\right)\cdot\dot{\phi} + \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\cdot\dot{\mathbf{u}}_{,x} + \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\cdot\dot{\mathbf{u}}_{,y} \tag{96}$$

Further, by means of Green's theorem, we have the following:

$$\begin{aligned}&\int_\Omega\left(\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\cdot\dot{\mathbf{u}}_{,x} + \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\cdot\dot{\mathbf{u}}_{,y}\right)\mathrm{d}x\,\mathrm{d}y\\ &\quad= -\int_A\left\{\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\right)\right\}\cdot\dot{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\int_{\partial\Omega}\left\{\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\cdot\dot{\mathbf{u}}n_x + \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\cdot\dot{\mathbf{u}}n_y\right\}\mathrm{d}\Gamma = 0\\ &n_x = \frac{\partial x}{\partial n}, \qquad n_y = \frac{\partial y}{\partial n}\end{aligned} \tag{97}$$

where n_x, n_y denote the directional cosines, respectively, and n stands for the coordinate of the outward unit vector normal to the boundary surface. In view of Eqs. (96) and (97), Eq. (94) can be given by

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_\Omega\Bigg\{&\left(\dot{\boldsymbol{\phi}} + \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}} - \frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\right) - p\mathbf{a}\right)\cdot\dot{\mathbf{u}}\\ &+\left(\frac{\partial\bar{\mathcal{H}}}{\partial\boldsymbol{\phi}} - \dot{\mathbf{u}}\right)\cdot\dot{\boldsymbol{\phi}}\Bigg\}\,\mathrm{d}x\,\mathrm{d}y + \int_{\partial\Omega}\left\{\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\cdot\dot{\mathbf{u}}n_x\right.\\ &\left.+\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\cdot\dot{\mathbf{u}}n_y\right\}\mathrm{d}\Gamma = 0\end{aligned} \tag{98}$$

which is the hybrid weighted residual form. Because the variations of the dependent variables are arbitrary, the governing equations for the Reissner–Mindlin plate in terms of the Hamiltonian density function are obtained as

$$\dot{\boldsymbol{\phi}} = -\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}} + \frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\right) + p(x,y)\mathbf{a}, \quad \dot{\mathbf{u}} = \frac{\partial\bar{\mathcal{H}}}{\partial\boldsymbol{\phi}} \tag{99}$$

subject to the natural boundary conditions:

$$\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\cdot\dot{\mathbf{u}}n_x + \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\cdot\dot{\mathbf{u}}n_y = 0 \tag{100}$$

Upon substitution of Eq. (91) into Eq. (99), we obtain

$$\begin{gathered}\frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} + \left[\frac{\partial\boldsymbol{\kappa}}{\partial\mathbf{u}}\right]^T\mathbf{M}_b + \left[\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{u}}\right]^T\mathbf{Q}_s - \frac{\partial\mathbf{A}_x}{\partial x} - \frac{\partial\mathbf{A}_y}{\partial y} - p(x,y)\mathbf{a} = 0\\ \dot{\mathbf{u}} - \boldsymbol{\rho}^{-1}\boldsymbol{\phi} = 0\end{gathered} \tag{101}$$

where

$$\begin{aligned}\mathbf{A}_x &= \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}} = \left[\frac{\partial\boldsymbol{\kappa}}{\partial\mathbf{u}_{,x}}\right]^T\mathbf{M}_b + \left[\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{u}_{,x}}\right]^T\mathbf{Q}_s\\ \mathbf{A}_y &= \frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}} = \left[\frac{\partial\boldsymbol{\kappa}}{\partial\mathbf{u}_{,y}}\right]^T\mathbf{M}_b + \left[\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{u}_{,y}}\right]^T\mathbf{Q}_s\\ \left[\frac{\partial\boldsymbol{\kappa}}{\partial\mathbf{u}}\right] &= \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \qquad \left[\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{u}}\right] = \begin{bmatrix}0&0&-1\\0&1&0\end{bmatrix}\\ \left[\frac{\partial\boldsymbol{\kappa}}{\partial\mathbf{u}_{,x}}\right] &= \begin{bmatrix}0&1&0\\0&0&0\\0&0&-1\end{bmatrix}, \qquad \left[\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{u}_{,x}}\right] = \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}\\ \left[\frac{\partial\boldsymbol{\kappa}}{\partial\mathbf{u}_{,y}}\right] &= \begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}, \qquad \left[\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{u}_{,y}}\right] = \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}\end{aligned} \tag{102}$$

Therefore, Eq. (80) can be recast into

$$\begin{aligned}&\begin{Bmatrix}\dot{\varphi}_1\\\dot{\varphi}_2\\\dot{\varphi}_3\end{Bmatrix} + \begin{Bmatrix}0\\Q_y\\-Q_x\end{Bmatrix} - \frac{\partial}{\partial x}\begin{Bmatrix}Q_y\\M_{xx}\\-M_{xy}\end{Bmatrix} - \frac{\partial}{\partial y}\begin{Bmatrix}Q_x\\M_{xy}\\-M_{yy}\end{Bmatrix}\\ &\quad - \begin{Bmatrix}p\\0\\0\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}\\ &\begin{Bmatrix}\dot{w}\\\dot{\theta}_x\\\dot{\theta}_y\end{Bmatrix} - \begin{Bmatrix}\dot{\varphi}_1/-\rho h\\\dot{\varphi}_2/-\rho I_p\\\dot{\varphi}_3/-\rho I_p\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}\end{aligned} \tag{103}$$

The first equation in Eq. (99) implies the following:

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t}+\frac{\partial\bar{\mathcal{H}}}{\partial w}-\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w_{,x}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\bar{\mathcal{H}}}{\partial w_{,y}}\right)-p(x,y)=0$$
$$\frac{\mathrm{d}\phi_2}{\mathrm{d}t}+\frac{\partial\bar{\mathcal{H}}}{\partial\theta_x}-\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\theta_{x,x}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\theta_{x,y}}\right)=0$$
$$\frac{\mathrm{d}\phi_3}{\mathrm{d}t}+\frac{\partial\bar{\mathcal{H}}}{\partial\theta_y}-\frac{\partial}{\partial x}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\theta_{y,x}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\bar{\mathcal{H}}}{\partial\theta_{y,y}}\right)=0 \tag{104}$$

which lead to

$$\rho h\ddot{w}-Ghk\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}+\theta_x\right)-Ghk\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}-\theta_y\right)-p(x,y)=0$$
$$\rho I_p\ddot{\theta}_x+Ghk\left(\frac{\partial w}{\partial x}+\theta_x\right)-\frac{1}{2}D\left\{(1-\nu)\nabla^2\theta_x+(1+\nu)\left(\frac{\partial^2\theta_x}{\partial x^2}-\frac{\partial^2\theta_y}{\partial y\partial x}\right)\right\}=0$$
$$\rho I_p\ddot{\theta}_y-Ghk\left(\frac{\partial w}{\partial y}-\theta_y\right)-\frac{1}{2}D\left\{(1-\nu)\nabla^2\theta_y+(1+\nu)\left(\frac{\partial^2\theta_y}{\partial y^2}-\frac{\partial^2\theta_x}{\partial x\partial y}\right)\right\}=0 \tag{105}$$

The second term in the right-hand side of Eq. (105) gives the natural boundary conditions as follows:

$$\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,x}}\cdot\dot{\mathbf{u}}n_x+\frac{\partial\bar{\mathcal{H}}}{\partial\mathbf{u}_{,y}}\cdot\dot{\mathbf{u}}n_y=n_x(Q_y\dot{w}+M_{xx}\dot{\theta}_x-M_{xy}\dot{\theta}_y)+n_y(Q_x\dot{w}+M_{xy}\dot{\theta}_x-M_{yy}\dot{\theta}_y)=\dot{w}(n_xQ_y+n_yQ_x)+\dot{\theta}_x(n_xM_{xx}+n_yM_{xy})-\dot{\theta}_y(n_xM_{xy}+n_yM_{yy})=0 \tag{106}$$

In view of Eq. (106), we say that the governing equations in Eq. (99) are subject to the natural boundary conditions on the mechanical boundary surface $\partial\Omega_\sigma$:

$$n_xQ_y+n_yQ_x=0,\qquad n_xM_{xx}+n_yM_{xy}=0$$
$$n_xM_{xy}+n_yM_{yy}=0 \tag{107}$$

Note that the essential boundary conditions on the geometric boundary surface $\partial\Omega_u$ can be given by

$$w=\bar{w},\qquad \theta_x=\bar{\theta}_x,\qquad \theta_y=\bar{\theta}_y \tag{108}$$

where $\bar{w}$, $\bar{\theta}_x$, and $\bar{\theta}_y$ are the prescribed displacement and rotation values. Subsequently, we observe that the rate of the Hamiltonian in Eq. (98) includes both governing equations and natural boundary conditions as the hybrid weighted residual form. Note that the exact solutions of the transverse deflection $w(x,t)$ the two rotations $\theta_x(x,y,t)$ and $\theta_y(x,y,t)$ should satisfy the continuity condition: namely, $w(x,y,t)$, $\theta_x(x,y,t)$, and $\theta_y(x,y,t)\in C^2(\Omega)$.

Next, we show that the proposed space-discrete Hamiltonian formulation readily facilitates the derivation of the space-discrete finite element equation of motion over Eq. (39). Instead of substituting the admissible functions into the hybrid weighted residual form in Eq. (98), we first construct the space-discrete autonomous Hamiltonian that allows the admissible trial functions: viz., $w^h(x,y,t)$, $\theta_x^h(x,y,t)$, and $\theta_y^h(x,y,t)\in H^1(\Omega)\subset C^0(\Omega)$. Thus, the Reissner–Mindlin element is often called a C^0 element. The admissible trial functions can be chosen as follows:

$$w^h(x,y,t)=\mathbf{H}_w(x,y)\mathbf{q}(t),\qquad \theta_x^h(x,t)=\mathbf{H}_{\theta_x}(x,y)\mathbf{q}(t)$$
$$\theta_y^h(x,t)=\mathbf{H}_{\theta_y}(x,y)\mathbf{q}(t)\qquad \mathbf{H}_w=[\,\mathbf{N}_w(x,y)\quad\mathbf{0}\quad\mathbf{0}\,]$$
$$\mathbf{H}_{\theta_x}=[\,\mathbf{0}\quad\mathbf{N}_{\theta_x}(x,y)\quad\mathbf{0}\,],\qquad \mathbf{H}_{\theta_y}=[\,\mathbf{0}\quad\mathbf{0}\quad\mathbf{N}_{\theta_y}(x,y)\,]$$
$$\mathbf{q}(t)=(w_1^h(t),w_2^h(t),\ldots,w_{n_{\text{node}}}^h(t),\theta_{x1}^h(t),\theta_{x2}^h(t),\ldots,\theta_{xn_{\text{node}}}^h(t),\theta_{y1}^h(t),\theta_{y2}^h(t),\ldots,\theta_{yn_{\text{node}}}^h(t))$$
$$\dot{\mathbf{q}}(t)=(\dot{w}_1^h(t),\dot{w}_2^h(t),\ldots,\dot{w}_{n_{\text{node}}}^h(t),\dot{\theta}_{x1}^h(t),\dot{\theta}_{x2}^h(t),\ldots,\dot{\theta}_{xn_{\text{node}}}^h(t),\dot{\theta}_{y1}^h(t),\dot{\theta}_{y2}^h(t),\ldots,\dot{\theta}_{yn_{\text{node}}}^h(t)) \tag{109}$$

The space-discrete Hamiltonian may be written as

$$\mathcal{H}^h(\mathbf{p},\mathbf{q})=\frac{1}{2}\mathbf{p}(t)\cdot\mathbf{M}^{-1}\mathbf{p}(t)+\frac{1}{2}\mathbf{q}(t)\cdot\mathbf{K}\mathbf{q}(t)-\mathbf{q}(t)\cdot\mathbf{F}_{\text{ext}}$$
$$\mathbf{p}(t)=\mathbf{M}\dot{\mathbf{q}}(t)\qquad \mathbf{M}=\int_A\mathbf{H}^T\boldsymbol{\rho}\mathbf{H}\,\mathrm{d}x\,\mathrm{d}y,\qquad \mathbf{H}=\begin{bmatrix}\mathbf{H}_w\\ \mathbf{H}_{\theta_x}\\ \mathbf{H}_{\theta_y}\end{bmatrix}$$
$$\mathbf{F}_{\text{ext}}=\int_\Omega p(x,y)\mathbf{H}_w^T\mathrm{d}x\mathrm{d}y\qquad \mathbf{K}=\mathbf{K}_b+\mathbf{K}_s$$
$$\mathbf{K}_b=\iint_\Omega\mathbf{B}_b^T\mathbf{D}_b\mathbf{B}_b\,\mathrm{d}x\,\mathrm{d}y,\qquad \mathbf{K}_s=\frac{k}{2}\iint_\Omega\mathbf{B}_s^T\mathbf{G}_s\mathbf{B}_s\,\mathrm{d}x\,\mathrm{d}y$$
$$\mathbf{B}_b^T=\begin{bmatrix}\frac{\partial\mathbf{H}_{\theta_x}}{\partial x} & -\frac{\partial\mathbf{H}_{\theta_y}}{\partial y} & (\frac{\partial\mathbf{H}_{\theta_x}}{\partial y}-\frac{\partial\mathbf{H}_{\theta_y}}{\partial x})\end{bmatrix}$$
$$\mathbf{B}_s^T=\begin{bmatrix}\frac{\partial\mathbf{H}_w}{\partial y}-\mathbf{N}_{\theta_y} & \frac{\partial\mathbf{H}_w}{\partial x}+\mathbf{N}_{\theta_x}\end{bmatrix} \tag{110}$$

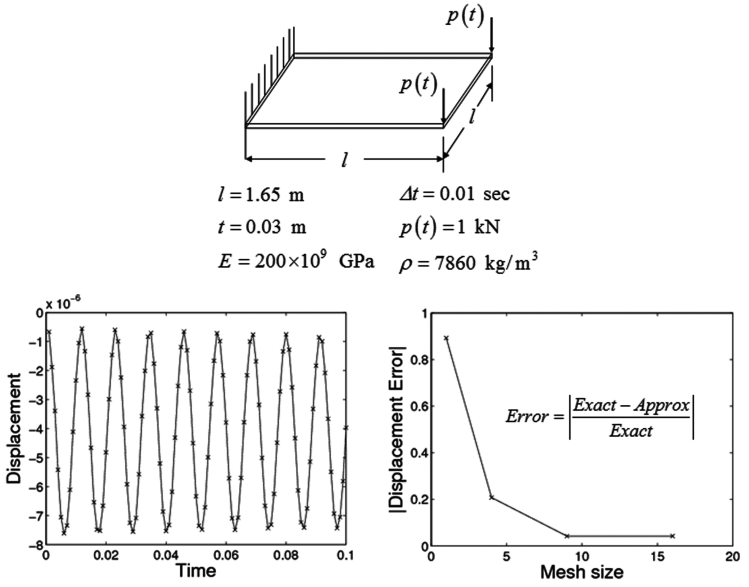

Fig. 2 Transient dynamical response of a Reissner–Mindlin plate.

Substituting the space-discrete Hamiltonian of Eq. (110) into the Hamiltonian differential operator given by Eq. (48), simply and naturally, leads to the semidiscrete finite element equations:

$$\mathfrak{S}_{D1}(\mathcal{H}^h) = \dot{\mathbf{p}} + \mathbf{K}\mathbf{q} - \mathbf{F}_{\text{ext}} = \mathbf{0}, \qquad \mathfrak{S}_{D2}(\mathcal{H}^h) = \dot{\mathbf{q}} - \mathbf{M}^{-1}\mathbf{p} = \mathbf{0} \tag{111}$$

A transient dynamical analysis of the Reissner–Mindlin plate is performed to provide a convergence plot with a uniform h refinement. Material properties, initial configuration, and loading conditions are depicted in Fig. 2. The exact solution is regarded as the solution obtained employing 64 elements (8 by 8). Thus, Fig. 2 shows the tip deflection with 64 elements and the time-step size of 0.001 s. Complete and compatible element and mesh are employed to demonstrate monotonic convergence with successive refinements. A uniform h refinement on the mesh size is carried out to describe the convergence plot, as shown in Fig. 2.

VI. Conclusions

Historically, the finite element method for elastodynamics applications has been explained in the sense of the principle of virtual work with the concept of variations from a certain viewpoint. Alternately, to circumvent the inherent disadvantages of traditional variational practices, alternative developments with a differential calculus were described with an improved physical interpretation and for ensuring that the discretized system also inherits the same physics as in the continuous system. Because of the inherent inconsistency of Hamilton's principle, as previously discussed in this work, we introduced and showed that the theorem of expended power involving the Hamiltonian density function is a more consistent and right alternative for deriving the governing equations with boundary conditions for scleronomic-dynamical systems of a continuous body. In addition, in contrast to traditional practices (which, after obtaining the model governing equations, seek to employ the weak form or principle of virtual work with trial and test functions to enact the discretization process), we proposed a space-discrete Hamiltonian finite element formulation involving only trial functions to directly discretize the theorem of expended power, without resorting to traditional practices, resulting in the space-discrete finite element equation of motion for continuum-dynamical systems. The illustrative examples of the Euler–Bernoulli beam and Reissner–Mindlin plate were demonstrated.

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